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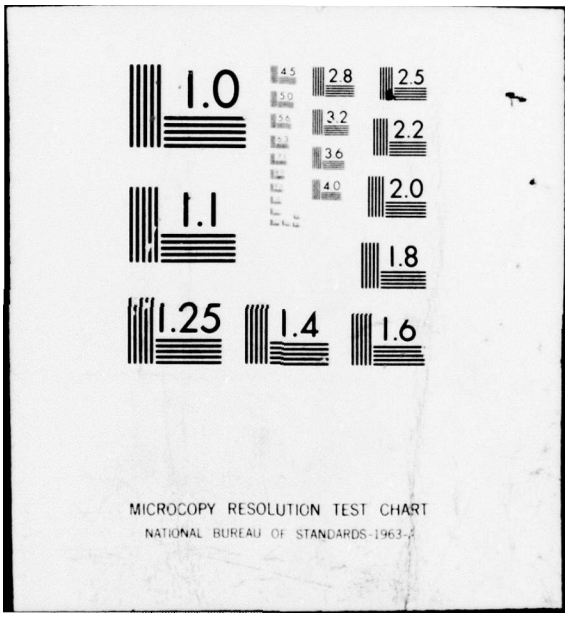
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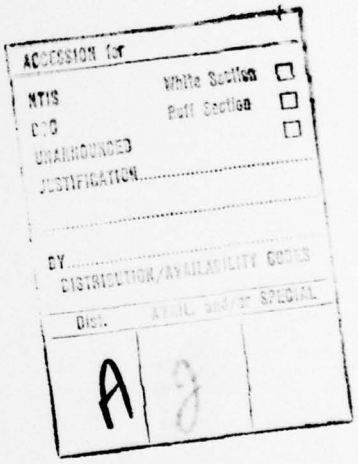

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ON RANDOM SEARCH WITH A LEARNING MEMORY

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A new class of random search algorithms for stochastic optimization is presented. The designer has the option to employ a learning memory in order to reduce the cost of the optimization process measured in terms of the number of observations. The asymptotical properties of the procedure are discussed, and new probability theoretical techniques are used in the proof of convergence.

I. Introduction

Let Q be an unknown real-valued function on a set $B \subset \mathbb{R}^m$ where $m \geq 1$. In many applications, one is interested in finding a w in B for which Q(w) is nearly minimal. Because of the absence of any information regarding the continuity, differentiability, smoothness or unimodality of Q, or because of the special nature of B (for example, B can be a countable set of isolated points from $\mathbb{R}^m$), it is not possible to use a classical optimization technique such as the gradient method. It is known that in such situations random search can be successfully used (for a review of the literature, see [1,2,3]).

In this paper, we are interested in the stochastic optimization problem, that is, Q(w) is no longer exactly computable but can be estimated if enough observations are averaged. To be explicit, it is assumed that for all $w \in B$, one can observe (compute, etc.) $Y_1, \ldots, Y_n, \ldots$ where the Y_n are independent random variables all distributed as Y with distribution function F_w and mean $Q(w) = \int y dF_w(y) = E_w\{Y\}$.

Several people have tried the random search algorithms used in deterministic optimization with the result that there are as many heuristic random search methods as there are scientists studying the stochastic optimization problem.

The most widely studied random search technique for stochastic optimization is the algorithm of Gurin [4] or one of its modifications [3,5]. Gurin's algorithm is simple and can be used for general B and Q. However, the task of proving the convergence for the modified methods has become increasingly difficult. Furthermore, Gurin's method is very inefficient with respect to the number of measurements (observations). If B is a finite set of points, one can use stochastic automata with a variable structure [6,7] or probabilistic strategy selection methods [8], most of which are proved to be convergent in some probabilistic sense. If Q satisfies some regularity conditions, usually in terms of continuity, differentiability and unimodality, local hill-climbing methods may be used. Most of these techniques are derived from the Kiefer-Wolfowitz stochastic approximation algorithm [10,11], the stochastic gradient algorithm [12-15] and combinations of these algorithms with stochastic automata and random search [16-17]. For instance, if Q is continuous and if the accuracy of the solution is of no great importance, one can always partition B into a finite number of sets and consider each set as a single point in a new space, thus reducing the problem to a finite optimization problem (see [9]).

The classical random search algorithm is a sequential procedure to update the best estimate of the minimum in which in the search for a new best estimate, only the very recent history of the search is taken into account. This algorithm thus operates with a short memory. However, over the last five years two factors in the design of optimization systems have changed. First, the computers have become very fast and can handle very large active memories. On the other hand, the cost of taking measurements (i.e. collecting data, evaluating a performance, etc.) has gone up considerably because of the increased cost of manpower. This has made the cost of the storage and processing of data decrease relatively to the cost of obtaining the data. This trend has been recognized by several authors (e.g. [18]). So, one wants to develop an algorithm which

(i) uses the available information as well as possible, e.g. by storing the past observations and processing the data obtained during the search in an intelligent way.

(ii) guarantees that the best estimate of the minimum converges, in some probabilistic sense, to the minimum of Q.

In this paper a statistical search method is developed with a potentially growing memory. The rate of convergence to the minimum is expected to be high due to the learning behavior of the memory. Maclaren [19] proposed, in a control engineering application, to use a stochastic automaton with a variable structure and a growing number of states to tackle a special stochastic optimization problem. However, the convergence problem for his method is not satisfactorily solved while the field of applications is very small. Our approach does not resemble any other method available in the literature and is partially modeled on the learning process in the human brain. "Remembering exceptional facts", "forgetting the too distant past" and "averaging costs" are features that can be recognized in the algorithm. The theoretical value of the method is that it encompasses the well-known random optimization method of Matyas [20] for deterministic optimization as a special case. The emphasis is on the new method for proving the convergence of the algorithm in stochastic optimization problems. The techniques, different from those employed in [4,5], depend upon some powerful probability theoretical inequalities [22].

II. Problem Formulation

Let $(\Omega, \mathcal{G}, P)$ be a probability space and let B be a closed set from $\mathbb{R}^m$. Let $\mathcal{B}_B^m$ be the σ-algebra of all the Borel sets that are contained in B. We assume that there exists a measurable mapping h from $(\Omega \times B, \mathcal{G} \times \mathcal{B}_B^m)$ to $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the class of Borel sets from $\mathbb{R}$. Notice that for every w in $\mathbb{R}^m$, $Y = h(\omega, w)$ is a random variable on $(\Omega, \mathcal{G}, P)$. We say that a collection

$$\mathcal{E} = \{F_w \mid w \in B\} \qquad (1)$$

of distribution functions is a random environment with search domain B if B is a closed set from $\mathbb{R}^m$ and if there exists a probability space $(\Omega, \mathcal{G}, P)$ and an $(\Omega \times B, \mathcal{G} \times \mathcal{B}_B^m)$-$(\mathbb{R}, \mathcal{B})$ measurable function h such

that for all $y \in R$, $F_w(y)=P\{\omega \mid \omega \in \Omega, h(\omega,w) \leq y\}$. Notice that if B is countable, then such a probability space and a measurable function h can always be found. Thus it makes sense to define a countably infinite (finite) random environment as a countable (finite) collection of distribution functions. The reason of the definition (1) is the following. If W is any random vector on some probability space $(\Omega', \mathcal{G}', P')$ that is different from $(\Omega, \mathcal{G}, P)$ and W takes values in B, then $Y=h(\omega, W)$ is a random variable on the product of both probability spaces. Furthermore, if $W_1, \ldots, W_n$ is any sequence of random vectors (all taking values in B) that are applied to the environment, then there exists a sequence $Y_1, \ldots, Y_n$ of random variables (referred to as responses of the environment, measurements, observations or observed losses) where, given that $W_1=w_1, \ldots, W_n=w_n$ $(w_i \in B, i=1, \ldots, n)$, the Y_i are independent and have distribution function F_{w_i}, $i=1, \ldots, n$.

We will refer to $Q(w)=\int y dF_w(y) = E_w\{Y\}$ as the stochastic performance index. Q is by assumption a Borel measurable function from B to R. Except for B, we assume that there is no a priori knowledge about $\mathcal{E}$ or Q. The stochastic optimization problem is to sequentially find a value w in B for which Q is minimal or nearly minimal.

We assume that there is a random generator with support in B, i.e. a device for generating a sequence $W_1, \ldots, W_n, \ldots$ of iid (independent identically distributed) random vectors taking values in B and distributed as W where W has a distribution function G which is either known or unknown. The minimum of Q with respect to G is

$$q_{min} = \text{ess inf } Q(W) \qquad (2)$$

where the essential infimum is defined as usual [2]. Actually, q_{min} is the unique number with the property that for all $\epsilon > 0$, $P\{Q(W) \leq q_{min} - \epsilon\} = 0$ and $P\{Q(W) \leq q_{min} + \epsilon\} > 0$ provided that $q_{min} > -\infty$. We remark that if B is countable, say $B=\{w_1, w_2, \ldots\}$, and G puts mass g_i at w_i such that $\Sigma g_i = 1$ and $0 \leq g_i \leq 1$ for all i, then $q_{min} = \inf_{i: g_i > 0} Q(w_i)$. In this case we see that q_{min} is independent of G as long as every w_i receives positive probability from G.

We categorize the random environments with search domain B as follows.

(i) $\mathcal{E}$ is $\mathcal{S}$ (deterministic, noiseless) if for all $w \in B$, $Y=Q(w)$ wpl (with probability one).

(ii) $\mathcal{E}$ is $\mathcal{E}_t$ for some $t>0$ with parameter $L<\infty$ if

$$\sup_{w \in B} E_w\{|Y-Q(w)|^t\} = \sup_{w \in B} \int |y-Q(w)|^t dF_w(y) \leq L < \infty \qquad (3)$$

(iii) $\mathcal{E}$ is $\mathcal{J}$ (generalized gaussian) with parameters σ and L $(0 \leq \sigma < \infty, 0 \leq L < \infty)$ if

$$\sup_{w \in B} E_w\{e^{\lambda(Y-Q(w))}\} \leq e^{\sigma^2 \lambda^2 / 2(1-|\lambda|L)}$$

for all λ with $|\lambda|L<1$.

If an environment is $\mathcal{J}$ then it certainly is $\mathcal{E}_t$ for all $t>0$. A deterministic environment is always gaussian and if $\mathcal{E}$ is $\mathcal{E}_t$ for some $t>0$ with parameter $L=0$, then $\mathcal{E}$ is deterministic. Also, if $\mathcal{E}$ is $\mathcal{E}_t$ then $\mathcal{E}$ is $\mathcal{E}_s$ for all $s \leq t$. It should be pointed out that most environments of any practical interest are generalized gaussian. For instance, if all the F_w are gaussian with variance $\sigma^2(w) \leq \sigma^2$, then $\mathcal{E}$ is $\mathcal{J}$ with parameters σ and 0. If to all F_w correspond probability measures that put their weights on $[d_1(w), d_2(w)]$ where $d_2(w)-d_1(w) \leq L$, then $\mathcal{E}$ is generalized gaussian with parameters $L/2$ and L. In particular, if for all w in B, Y takes with probability one values in $\{0,1\}$ or $[0,1]$, then $\mathcal{E}$ is $\mathcal{J}$ with parameters $1/2$ and 1. Such environments are often encountered in stochastic automata theory and discrete optimization.

The purpose is to find an optimization procedure that generates a sequence of random vectors $W_1, \ldots, W_n, \ldots$ taking values in B such that $\text{Max}(Q(W_n), q_{min})$ tends in some probabilistic sense to q_{min} as $n \to \infty$. Notice that we have to allow for the possibility that $Q(W_n) < q_{min}$ for some n. Of course, if $W_1, \ldots, W_n, \ldots$ were a sequence of iid random vectors distributed as W, then $Q(W_n) \geq q_{min}$ wpl for all n.

III. The Optimization Procedure

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences from $[0,1]$ with $\alpha_n + \beta_n \leq 1$ for all n, and let $\{\lambda_n\}$ be a sequence of positive integers. Further, let $Z_1, Z_2, \ldots$ be a sequence of independent integer-valued random variables with

$$P\{Z_n=1\}=\alpha_n, \; P\{Z_n=0\}=\beta_n, \; P\{Z_n=-1\}=1-\alpha_n-\beta_n. \qquad (4)$$

To start the search, generate a random vector $W_0^*=W_0$ having distribution function G. Given that $W_0^*=w$, λ_0 measurements are made, say $Y_1', \ldots, Y_{\lambda_0}'$, all having distribution function F_w. Let the estimate of $Q(w)$ be Y_0^* where

$$Y_0^* = \left(\sum_{i=1,}^{\lambda_0} Y_i'\right)/\lambda_0 . \qquad (5)$$

Let $Y_0=Y_0^*$ and $N_0=N_0^*=\lambda_0$, where N_0 is the number of observations that were used in computing the average Y_0. The search procedure consists of generating two sequences of triples, (W_n^*, Y_n^*, N_n^*) and (W_n, Y_n, N_n), $n=1,2,\ldots$ where $(W_0^*, Y_0^*, N_0^*)=(W_0, Y_0, N_0)$. W_n is the estimate at iteration n of the minimum in R^m of Q. Y_n is the corresponding estimate of $Q(W_n)$ and N_n is the experience with W_n, that is, the number of observations that were used in the computation of the estimate Y_n.

Let the search be at iteration n. Then W_n^* is generated as follows.

(i) If $Z_n=0$, let $W_n^*=W_{n-1}$.

(ii) If $Z_n=1$, let W_n^* be an independent random vector with distribution function G.

(iii) If $Z_n=-1$, W_n^* is arbitrary with the restriction that $P\{W^* \in B\}=1$.

Given that $W_n^*=w$, $N_n^*=\lambda_n$ observations are made, say $Y_1', \ldots, Y_{\lambda_n}'$, all having distribution function F_w. Let the estimate of $Q(w)$ be

$$Y_n^* = \left(\sum_{i=1}^{\lambda_n} Y_i'\right)/\lambda_n . \tag{6}$$

Two questions naturally arise; (1) How does one pick W_n^* if $Z_n=-1$ and how can the past observations be used to aid in picking W_n^*? (2) How does one find (W_n, Y_n, N_n) given (W_n^*, Y_n^*, N_n^*), $i=0,1,\ldots,n$ and $(W_{n-1}, Y_{n-1}, N_{n-1})$? We will refer to W_n as the basepoint and to (W_n, Y_n, N_n) as the base triple.

In Gurin's algorithm [3-5], to make the decision in answer to question (2), it is required that λ_n additional measurements be made with W_{n-1} to obtain an estimate $\tilde{Y}_{n-1}$ of $Q(W_{n-1})$. The decision is based upon a comparison between $\tilde{Y}_{n-1}$ and Y_n^*, that is, $W_n=W_{n-1}$ unless $\tilde{Y}_{n-1}>Y_n^*+\epsilon_n$ (where $\epsilon_n>0$ is a threshold), in which case $W_n=W_n^*$. However, valuable data are wasted since Y_{n-1} is forgotten and thus, it is as if N_{n-1} measurements are thrown away at the n-th iteration. Therefore, we will not require to make special additional observations for the decision (2), thus reducing the total cost of data collection.

Let H_n be the data, outside the base triple, that are memorized at time n where $H_n=(W_1^n, Y_1^n, N_1^n), \ldots, (W_{T_n}^n, Y_{T_n}^n, N_{T_n}^n)$ and where T_n is a nonnegative integer valued random variable. If $T_n=0$, then H_n is empty. If $T_n \le M<\infty$ for all n, we say that the algorithm operates with a finite memory. If $T_n \to \infty$ as $n\to\infty$ then we say that the algorithm operates with a growing memory. We require that H_n be a measurable function of the (W_i^*, Y_i^*, N_i^*), $i=0,1,\ldots,n$ and that at all times $W_n, W_1^n, \ldots, W_{T_n}^n$ are pairwise unequal. Therefore, $T_0=0$ and $T_n \le n$ for all n.

We now continue the description of the algorithm. First of all, it is clear that in picking W_n^* if $Z_n=-1$, we can expect help from H_{n-1} and $(W_{n-1}, Y_{n-1}, N_{n-1})$. Given (W_n^*, Y_n^*, N_n^*), $(W_{n-1}, Y_{n-1}, N_{n-1})$ and H_{n-1}, we will compute (W_n, Y_n, N_n) in two steps. First an auxiliary triple $(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)$ is obtained. Define a random variable S_n where

$$\begin{cases} S_n=1 \text{ if } W_n^*=W_{n-1} \\ S_n=2 \text{ if } W_n^*=W_i^{n-1} \text{ for some } W_i^{n-1} \text{ from } H_{n-1} \\ S_n=3 \text{ otherwise} \end{cases} \tag{7}$$

Note that S_n is uniquely defined since $W_{n-1}, W_1^{n-1}, \ldots, W_{T_{n-1}}^{n-1}$ are pairwise unequal. Define further the merging of two triples, (W,Y,N) and $(W,Y",N")$, as follows:

$$(W,Y,N)*(W,Y",N")=(W,(NY+N"Y")/(N+N"),N+N") \tag{8}$$

Thus, the experience of the new triple is the sum of the experiences of the component triples.

Define $(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)$ by

$$(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)=\begin{cases} (W_n^*, Y_n^*, N_n^*) \text{ or } (W_n^*, Y_n^*, N_n^*)* \\ \quad (W_{n-1}, Y_{n-1}, N_{n-1}), \text{ if } S_n=1 \\ (W_n^*, Y_n^*, N_n^*) \text{ or } (W_n^*, Y_n^*, N_n^*)* \\ \quad (W_i^{n-1}, Y_i^{n-1}, N_i^{n-1}), \\ \quad \text{if } S_n=2 \text{ and } W_n^*=W_i^{n-1} \\ (W_n^*, Y_n^*, N_n^*), \text{ if } S_n=3 \end{cases} \tag{9}$$

where, if $S_n=1$, one either always merges or never merges and, if $S_n=2$, one either always merges or never merges. The merging operation can be randomized but this will only complicate matters now. The consistency in the use of the merging operation and the fact that W_{n-1} and the W_i^{n-1}, $1\le i\le T_{n-1}$, are pairwise unequal for all n are important factors in the proof of the theorem of convergence given below.

The next step is the decision whether to pick $(W_{n-1}, Y_{n-1}, N_{n-1})$ or to select $(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)$ as the new base triple. Let D_n be a random variable taking values in $\{0,1\}$ where $D_n=1$ only if the old base triple is updated at the n-th iteration. Thus,

$$D_n=\begin{cases} 1 \text{ if } S_n=1 \text{ or } \bar{Y}_n^*<Y_{n-1} \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

and

$$(W_n, Y_n, N_n)=\begin{cases} (W_n^*, \bar{Y}_n^*, \bar{N}_n^*) & \text{if } D_n=1 \\ (W_{n-1}, Y_{n-1}, N_{n-1}) & \text{if } D_n=0. \end{cases} \tag{11}$$

The only thing that is left is to obtain H_n from H_{n-1} and (W_n^*, Y_n^*, N_n^*). To make sure that W_n^n and W_i^n, $1\le i\le T_n$, are pairwise unequal, the following procedure is suggested.

(i) If $S_n=2$ and $W_n^*=W_i^{n-1}$, remove $(W_i^{n-1}, Y_i^{n-1}, N_i^{n-1})$ from H_{n-1}.

(ii) If $D_n=1$ and $S_n\neq 1$, add $(W_{n-1}, Y_{n-1}, N_{n-1})$ to H_{n-1} or add nothing at all. If $D_n=0$, add $(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)$ to H_{n-1} or add nothing at all.

(iii) Any triple left in H_{n-1} after (ii) can be dropped if desired. Dropping triples corresponds to a loss of memory but can sometimes be more economical.

(iv) Relabel all the triples left after (iii) so that to all $1\le i\le T_n$ (T_n is the number of triples) there corresponds one and only one triple (W_i^n, Y_i^n, N_i^n). This relabeled sequence of triples is H_n.

The method of deciding whether to add or to drop triples from H_{n-1} (in (ii), (iii)) is not specified. In fact this decision may depend in an arbitrary fashion upon any information available at the n-th iteration. The decision may be randomized and can, in an extreme case, also be made through human intervention in the search process.

Given (W_n, Y_n, N_n) and H_n, the above described procedure is repeated for n+1, that is, the generation of $(W_{n+1}^*, Y_{n+1}^*, N_{n+1}^*)$ (see (6)), the computation of $(W_{n+1}^*, \bar{Y}_{n+1}^*, \bar{N}_{n+1}^*)$ (see (7), (9)), the decision concerning $(W_{n+1}, Y_{n+1}, N_{n+1})$ (see (10-11)) and the determination

of H_{n+1} (procedure (i)-(iv)).

Remarks : We note that the algorithm can be used with $T_n=0$ for all n. Notice further that $T_0=0$ and that $0 \le T_{n+1} \le T_n+1$ for all n. The memory can be labeled as a learning memory either because the N_n^* are increasing (in view of $\lambda_n \to \infty$ as $n \to \infty$, or because of the merging in (9)) or because $T_n \to \infty$ as $n \to \infty$. Surprisingly, the convergence of the algorithm is not affected by the finiteness or divergence of the sequence T_n.

The undefined parts in the algorithm are
(a) The generation of W_n^* if $Z_n=-1$.
(b) Steps (ii) and (iii) in the updating of H_{n-1}.

It is up to the designer to use (a) and (b) to obtain high rates of convergence. Of course, some experimental know-how will be helpful. Let us briefly discuss the problems (a) and (b). We say that Q and B define an exhaustive search problem if for every finite subset $\{w_1, \ldots, w_L\}$ of B, the knowledge of $Q(w_1), \ldots, Q(w_L)$ does not convey any information regarding the value of Q(w) for any $w \in B, w \notin \{w_1, \ldots, w_L\}$. In such problems, does it still make sense to store some information in H_n (i.e., to let $T_n>0$) ? The answer is of course negative if the environment is deterministic. Indeed, if the environment is $\mathcal{S}$, then it is clear that $Y_n=Q(W_n)$ wp1 for all n. The only information that needs to be stored is (W_n, Y_n) and it is not necessary to sample the basepoint (thus, let $\beta_n=0$ for all n). Further, if $Z_n=-1$, the best one can do is to generate W_n^* with distribution function G in B. Therefore, if the environment is deterministic and defines an exhaustive search problem, we can let $\alpha_n=1$ and $Z_n=1$ for all n. In the random search literature, this method is called blind search [1]. Assume next that the environment is deterministic but that Q and B do not define an exhaustive search problem, e.g. because $B=\mathbb{R}^m$ and Q is known to be continuous. In that case it can be helpful to let $T_n>0$. If $Z_n=-1, T_{n-1}=0$ and Q is continuous, one can let W_n^* be gaussian with variance σ_n^2 and mean W_{n-1} for the purpose of local hill-climbing (this method is referred to as creeping random search [1]). If $T_{n-1}>0$, the distribution of W_n^* may be a mixture of gaussian distributions with centers at W_{n-1} and $W_i^{n-1}, 1 \le i \le T_{n-1}$, in order to simultaneously climb separate local hills. The same strategies can be used if the environment is noisy, i.e. not $\mathcal{S}$. But for noisy environments, even in case Q and B define an exhaustive search problem, it makes sense to store all the past observations in H_n on account of the fact that the Y_i^{n-1} are only noisy estimates of the $Q(W_i^{n-1})$ for $i=1, \ldots, T_{n-1}$. In such case, if $Z_n=-1$, one could for instance define W_n^* as follows. Let $M>0$ be fixed and consider those W_i^{n-1} that correspond to the M lowest values among the $Y_i^{n-1}, 1 \le i \le T_{n-1}$. Then let W_n^* have a uniform distribution over those W_i^{n-1}. The designer can for instance eliminate the other $(W_i^{n-1}, Y_i^{n-1}, N_i^{n-1})$ from H_{n-1} so that $T_n \le M$ for all n.

A note is in order here concerning the merging operation in (9). If T_{n-1} is large and one goes through the trouble of storing all or most of the past observations, it would be very inconsistent if no merging was used in (9). Further, if merging is used in (9), it is wise to let W_n^* be equal to one of the $W_i^{n-1}, 1 \le i \le T_{n-1}$, with positive probability, thus increasing the experiences of the W_i^{n-1} on the long run. If T_n is small or zero, one can of course as well do without the merging in (9). This would simplify the algorithm considerably because $(W_n^*, \bar{Y}_n^*, \bar{N}_n^*)=(W_n^*, Y_n^*, N_n^*)$ and $N_n^* \equiv \lambda_n$. If $T_{n-1}=0$ for all n, then it is easy to see that the only thing to be memorized is (W_{n-1}, Y_{n-1}). The decision rule (10-11) reduces to

$$(W_n, Y_n) = \begin{cases} (W_n^*, Y_n^*) & \text{if } Y_n^* < Y_{n-1} \\ (W_{n-1}, Y_{n-1}) & \text{otherwise.} \end{cases} \tag{12}$$

In that case the algorithm reduces to the well-known random optimization algorithm of Matyas [20].

IV. Theorem Of Convergence

Theorem 1 : Let B be a closed set from $\mathbb{R}^m$ and let $\mathcal{E}$ be a random environment with search domain B. Let Q be a Borel measurable mapping from B to $\mathbb{R}$ and let G be an arbitrary distribution function with support in B. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be number sequences from $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$ for all n. Let $\{\lambda_n\}$ be a sequence of positive integers and let $W_1, W_2, \ldots$ be a sequence of random vectors from $\mathbb{R}^m$ whose distribution is determined by the procedure described in section III. If there exists a sequence $\{b_n\}$ of integers such that

$$b_0 \le b_1 \le \ldots \tag{13}$$

$$0 \le b_n \le n \text{ for all } n, \tag{14}$$

$$\sum_{i=b_n}^{n} \beta_i \to \infty, \tag{15}$$

$$\sum_{i=b_n}^{n} \alpha_i \to \infty, \tag{16}$$

and the environment is

either $\mathcal{S}$
or $\mathcal{J}$ and $\bar{\lambda}_{b_n}/c_n^2 \log n \to \infty$

or $\mathcal{E}_t$ for $t \ge 2$ and in addition to the latter condition, $n c_n^t / \bar{\lambda}_{b_n}^{t-1} \to \infty$ (17)

where $c_n=n-b_n+1$ and $\bar{\lambda}_{b_n} \triangleq \text{Min}(\lambda_{b_n}, \lambda_{b_n+1}, \ldots, \lambda_n)$, then

$$\text{Max}(Q(W_n), q_{min}) \to q_{min} \text{ in probability.} \tag{18}$$

The convergence in (18) is with probability one if the conditions (15-17) are replaced by (15'-17'):

$$\sum_{i=b_n}^{n} \beta_i / \log n \to \infty, \tag{15'}$$

$$\sum_{i=b_n}^{n} \alpha_i / \log n \to \infty, \tag{16'}$$

and the environment is either $\mathcal{S}$ or $\mathcal{J}$ and $\bar{\lambda}_{b_n}/c_n^2 \log n \to \infty$, or $\mathcal{E}_t$ for $t \ge 2$ and in addition to the latter condition, $\sum_{n=1}^{\infty} n c_n^t / \bar{\lambda}_{b_n}^{t-1} < \infty$. (17')

Proof: Theorem 1 is proved in the Appendix.

In some applications one is more interested in the asymptotic behavior of the expected values of the measurements, i.e. $Q(W_n^*)$, $n=1,2,\ldots$. The following theorem holds true.

Theorem 2: Let B be a closed set from R^m and let $\mathcal{E}$ be a random environment with search domain B. Let Q be a Borel measurable mapping from B to R and let G be an arbitrary distribution function with support in B. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be number sequences from $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$ for all n. Let $\{\lambda_n\}$ be a sequence of positive integers and let $W_1^*, W_1, W_2^*, W_2, \ldots$ be a sequence of random vectors from R^m whose distribution is determined by the procedure described in section III. If

$$\mathrm{Max}(Q(W_n), q_{min}) \xrightarrow{n} q_{min} \text{ in probability} \qquad (19)$$

and

$$\beta_n \xrightarrow{n} 1, \qquad (20)$$

then

$$\mathrm{Max}(Q(W_n^*), q_{min}) \xrightarrow{n} q_{min} \text{ in probability.} \qquad (21)$$

Proof: Let $\epsilon>0$ be arbitrary and note that $\{Q(W_n^*)>q_{min}+\epsilon\} \subseteq \{Q(W_{n-1})>q_{min}+\epsilon\} \cup \{Z_n \neq 0\}$, so that $P\{Q(W_n^*)>q_{min}+\epsilon\} \le P\{Q(W_{n-1})>q_{min}+\epsilon\}+(1-\beta_n)$. Theorem 2 follows from this inequality, (19) and (20).

Remark: It turns out that the convergence in probability of $Q(W_n^*)$, as in (21), normally is the strongest possible mode of convergence. Indeed, if $\gamma_n=0$ for all n, it is not always possible to insure that $Q(W_n^*)$ converges wp1. This curious but not entirely surprising result is formulated in Theorem 3. The counterexample proving Theorem 3 is given in the Appendix. The result in Theorem 3 is not absolute in the sense that for special B and $\mathcal{E}$ it may be possible that $Q(W_n)$ and $Q(W_n^*)$ both tend to q_{min} wp1 as $n\to\infty$.

Theorem 3: There exists a closed set B from R^m, a Borel measurable function Q from B to R, a deterministic environment $\mathcal{E}$ and a distribution function G with support in B such that for all sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ from $[0,1]$ with $\alpha_n+\beta_n=1$ and $\gamma_n=0$ and for all sequences $\{\lambda_n\}$ of positive integers and for all algorithms fitting the description of section III, it is impossible that

$$\mathrm{Max}(Q(W_n^*), q_{min}) \xrightarrow{n} q_{min} \text{ wp1.}$$

For deterministic environments, one can let $b_n=1$ in the conditions of Theorem 1. The conditions of convergence then reduce to

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n = \infty.$$

By a slight change in the proof of the theorem, it can be seen that the condition $\Sigma\beta_n=\infty$ can be dropped altogether.

The conditions of convergence in Theorem 1 look rather complicated. Let for instance $\alpha_n=A/n^{\alpha}$, $\beta_n=B/n^{\beta}$ and $\lambda_n=Cn^{\delta}$ where $\alpha\ge 0$, $\beta\ge 0$ and $\delta\ge 0$. If the environment is $\mathcal{E}_t$ with $t\ge 2$, then (13-17) hold if

$$\mathrm{Max}(\alpha,\beta)<\mathrm{Min}(\delta/2,(\delta(t-1)-1)/t,\ 1) \qquad (22)$$

and (13-14, 15'-17') hold if

$$\mathrm{Max}(\alpha,\beta)<\mathrm{Min}(\delta/2,(\delta(t-1)-2)/t,\ 1). \qquad (23)$$

If the environment is $\mathcal{S}$, then (13-17) or (13-14, 15'-17') hold if

$$\mathrm{Max}(\alpha,\beta)<\mathrm{Min}(\delta/2,1). \qquad (24)$$

For this, it suffices that $\alpha=\beta=0$ and that $\delta>0$. The proofs of the sufficiency of (22-24) are given in the Appendix.

V. Conclusion

The theoretical properties of a large class of random search algorithms for use in stochastic optimization are discussed. To actually obtain practical algorithms, it is important to make the best use of the freedom that is left to the designer, e.g. in the choice of the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$, in the procedure for the generation of W_n^* and in the procedure for updating the memory contents H_n. As for most random search techniques, the class of random environments to be allowed is very large. This makes the algorithm suitable as a basic building block for a widely applicable optimization program in the computer library.

The designer has the option to use an algorithm with a growing memory to reduce the cost of optimization measured in terms of the number of observations. It is pointed out how a growing memory can be useful even in exhaustive (but stochastic) optimization problems. In non-exhaustive search problems, e.g. when $B=R^m$ and Q is continuous, other procedures to extract information from the past observations should be studied. For instance, further research is incouraged in parametric and nonparametric estimators of Q that use the data that are collected during the search.

VI. Appendix

Lemma 1: Let $X_1,\ldots,X_n,X_1',\ldots,X_n'$ be iid random variables with $E\{X_1\}=0$ and $E\{X_1^2\}=\sigma^2<\infty$. If $S_n=\sum_{i=1}^{n} X_i$ and $S_n'=\sum_{i=1}^{n}(X_i-X_i')$, then

$$P\{\bigcup_{k\ge n}\{|S_k/k|\ge\epsilon\}\} \le 6 \sum_{k\ge \log_2 n} P\{|S'_{2^{k-1}}/2^k|\ge\epsilon/8\}$$

for all n and $\epsilon>0$ with $n\epsilon^2>8\sigma^2$.

Proof: Let μY denote the median of a random variable Y. By P. Levy's symmetrization inequality and the fact that if $E\{Y\}=0$, then $|\mu Y| \le (2E\{Y^2\})^{\frac{1}{2}}$,

$$P\{\bigcup_{k\ge n}\{|S_k/k|\ge\epsilon\}\} \le P\{\bigcup_{k\ge n}\{|(S_k-\mu S_k)/k|\ge\epsilon/2\}\} + P\{\bigcup_{k\ge n}\{|\mu S_k/k|\ge\epsilon/2\}\} \le$$

$$2P\{\bigcup_{k\ge n}\{|S_k'/k|\ge\epsilon/2\}\}+P\{\bigcup_{k\ge n}\{(2\sigma^2 k)^{\frac{1}{2}}/k\ge\epsilon/2\}\}.$$

The last term on the right-hand side of this inequality is 0 if $n\epsilon^2/4>2\sigma^2$. Arguing as in Loeve [21, pp. 252-253], we have for $2^{k-1}<n\le 2^k$,

$$|S_n'/n| = |(S_n'-S'_{2^{k-1}})/n + S'_{2^{k-1}}/n|$$

$$\le |S_n'-S'_{2^{k-1}}|/n + |S'_{2^{k-1}}|/2^{k-1}$$

$$\le 2(|S_n'-S'_{2^{k-1}}|/2^k + |S'_{2^{k-1}}|/2^k).$$

By another application of Levy's symmetrization inequality, $P\{\bigcup_{j\ge n}\{|S_j'/j|\ge\epsilon/2\}\}$

$$\le \sum_{2^k \ge n} \left(P\{\bigcup_{j=2^{k-1}+1}^{2^k} \{2|S'_j - S'_{2^{k-1}}|/2^k \ge \epsilon/4\}\} + P\{2|S'_{2^{k-1}}|/2^k \ge \epsilon/4\}\right)$$

$$\le \sum_{2^k \ge n} \left(2P\{|S'_{2^k} - S'_{2^{k-1}}|/2^k \ge \epsilon/8\} + P\{|S'_{2^{k-1}}|/2^k \ge \epsilon/8\}\right)$$

$$= 3 \sum_{k \ge \log_2 n} P\{|S'_{2^{k-1}}|/2^k \ge \epsilon/8\}.$$

Lemma 2 :Let $X_1, \ldots, X_n$ be iid random variables with $E\{X_1\}=0$ and $E\{|X_1|^t\} \le L < \infty$ for some $t \ge 2$, then

$$P\{\bigcup_{k \ge n} \{|S_k|/k \ge \epsilon\}\} \le C_1/\epsilon^t n^{t-1} + C_2 e^{-C_3 n \epsilon^2}$$

for all n and $\epsilon > 0$ with $n\epsilon^2 > 8L^{2/t}$ where $C_1 = 24(1+2/t)^t L 8^t$, $C_2 = 12/(1-\exp(-16e^2))$ and $C_3 = 1/(32e^t(2+t)L^{2/t})$.

Proof : From lemma 1, an inequality of Fuk and Nagaev [22, pp. 654] and $E\{X_1^2\} = \sigma^2 \le L^{2/t}$, we have with $S'_n = X_1 + \ldots + X_n - X_{n+1} - \ldots - X_{2n}$,

$$P\{\bigcup_{k \ge n} \{|S_k|/k \ge \epsilon\}\} \le 6 \sum_{k \ge \log_2 n} P\{|S'_{2^{k-1}}|/2^k \ge \epsilon/8\}$$

$$\le 6 \sum_{k \ge \log_2 n} 2(1+2/t)^t L/((\epsilon/8)^t (2^{t-1})^k) + 2\exp(-2e^t 2^k (\epsilon/8)^2/((t+2)\sigma^2))$$

$$\le (C_1/2\epsilon^t n^{t-1}) \sum_{k=0}^{\infty} (2^{t-1})^{-k} + C_2(1-\exp(-1/16e^2)) \sum_{k \ge \log_2 n} \exp(-C_3 \epsilon^2 2^k)$$

$$\le C_1/\epsilon^t n^{t-1} + (e^{-C_3 n \epsilon^2}/(1-e^{-C_3 n \epsilon^2}))C_2(1-e^{-1/16e^2})$$

$$\le C_1/\epsilon^t n^{t-1} + C_2 e^{-C_3 n \epsilon^2}$$

for all n with $n\epsilon^2 > 8L^{2/t}$ in view of $e^{-C_3 n \epsilon^2} \le e^{-C_3 8 L^{2/t}} \le e^{-1/16e^2}$. We used the fact that for all $a>1$, $b>1$ and K integer, $\sum_{k \ge K} a^{-b^k} \le a^{-b^K}/(1-a^{-b^K})$.

Lemma 3 :Let $X_1, \ldots, X_n$ be iid random variables with $E\{X_1\}=0$ and $E\{e^{\lambda X_1}\} \le e^{\lambda^2 \sigma^2/2(1-|\lambda|L)}$ for all λ with $|\lambda|L<1$ and for some $\sigma \ge 0$ and $L \ge 0$, then

$$P\{\bigcup_{k \ge n} \{|S_k/k| \ge \epsilon\}\} \le C_4 \exp(-n\epsilon^2/(128\sigma^2 + 16L\epsilon))$$

for all n and $\epsilon > 0$ with $n\epsilon^2 > 8\sigma^2$, where $C_4 = 12/(1-\exp(-1/16(1+L\epsilon/8\sigma^2)))$.

Proof :It is easy to see that $E\{X_1^2\} = \sigma^2$. Note also that for all n, by Chebyshev's inequality,

$$P\{S_n/n \ge \epsilon\} \le P\{\sum_{i=1}^n X_i \ge n\epsilon\} \le e^{-\lambda n \epsilon} (E\{e^{\lambda X_1}\})^n$$

$$\le \exp(-\lambda n \epsilon + n\lambda^2 \sigma^2/2(1-|\lambda|L)) \text{ for all } |\lambda|L<1.$$

With $\lambda L = \epsilon L/\sigma^2(1+L\epsilon/\sigma^2)$, we obtain

$$P\{S_n/n \ge \epsilon\} \le \exp(-n\epsilon^2/2\sigma^2(1+\epsilon L/\sigma^2)).$$

The same bound is valid for $P\{S_n/n \le -\epsilon\}$, so that by a combination of bounds,

$$P\{|S_n/n| \ge \epsilon\} \le 2\exp(-n\epsilon^2/2(\sigma^2 + L\epsilon)).$$

By lemma 1, for all n with $n\epsilon^2 > 8\sigma^2$,

$$P\{\bigcup_{k \ge n} \{|S_k/k| \ge \epsilon\}\} \le 12 \sum_{k \ge \log_2 n} \exp(-2^k(\epsilon/8)^2/2(\sigma^2 + L\epsilon/8))$$

$$\le 12 \exp(-n(\epsilon/8)^2/2(\sigma^2 + L\epsilon/8))/(1-e^{-n(\epsilon/8)^2/2(\sigma^2+L\epsilon/8)})$$

$$\le C_4 \exp(-n\epsilon^2/(128\sigma^2 + 16L\epsilon)).$$

Proof of theorem 1 :Let $\epsilon > 0$ be arbitrary and let $\{b_n\}$ be a sequence of integers satisfying (13-17). If $c_n = n - b_n + 1$, and W is a random vector with distribution function G, then we make the following crucial observation where we use $I_{\{.\}}$ to denote the indicator of an event.

$$\{Q(W_n) > q_{min} + \epsilon\} \subset \{\sum_{i=b_{n'}}^{n} I_{\{Z_i=0\}} = 0\} \cup \{\sum_{i=b_{n'}}^{n} I_{\{Z_i=0\}} \ge 1\} \cap \{\bigcup_{i=b_{n'}}^{n} \{|\bar{Y}_i^* - Q(W_i^*)| > \epsilon/4c_n\} \cup \{\bigcap_{i=b_{n'}}^{n} \{Q(W_i^*) > q_{min} + \epsilon/2\}\}\}. \quad (25)$$

First, it is clear that

$$P\{\sum_{i=b_{n'}}^{n} I_{\{Z_i=0\}} = 0\} \le \prod_{i=b_{n'}}^{n} (1-\beta_i) \le \exp(-\sum_{i=b_{n'}}^{n} \beta_i). \quad (26)$$

Also,

$$P\{\bigcap_{i=b_{n'}}^{n} \{Q(W_i^*) > q_{min} + \epsilon/2\}\} \le P\{\sum_{i=b_{n'}}^{n} I_{\{Z_i=1\}} \le \sum_{i=b_{n'}}^{n} \alpha_i/2\} + P\{\sum_{i=b_{n'}}^{n} I_{\{Z_i=1\}} > \sum_{i=b_{n'}}^{n} \alpha_i/2 ; \bigcap_{i=b_{n'}}^{n} \{Q(W_i^*) > q_{min} + \epsilon/2\}\}$$

$$\le (P\{Q(W) > q_{min} + \epsilon/2\})^{\sum_{i=b_{n'}}^{n} \alpha_i/2} + P\{\sum_{i=b_{n'}}^{n} (I_{\{Z_i=1\}} - \alpha_i)/c_n \le -\sum_{i=b_{n'}}^{n} \alpha_i/2c_n\}.$$

Using Bennett's inequality (see, e.g. [22]) and the fact that by the definition of q_{min}, $P\{Q(W) > q_{min} + \epsilon/2\} = 1-\theta$ for some $\theta > 0$, the right hand side of the last inequality is upper bounded by

$$(1-\theta)^{\sum_{i=b_{n'}}^{n} \alpha_i/2} + \exp(-c_n(\sum_{i=b_{n'}}^{n} \alpha_i/2c_n)^2/(2\sigma^2 + \sum_{i=b_{n'}}^{n} \alpha_i/2c_n))$$

where

$$\sigma^2 = \sum_{i=b_{n'}}^{n} \alpha_i(1-\alpha_i)/c_n \le \sum_{i=b_{n'}}^{n} \alpha_i/c_n.$$

Therefore, we can conclude that

$$P\{\bigcap_{i=b_{n'}}^{n} \{Q(W_i^*) > q_{min} + \epsilon/2\}\} \le \exp(-(\theta/2)\sum_{i=b_{n'}}^{n} \alpha_i) + \exp(-\sum_{i=b_{n'}}^{n} \alpha_i/10). \quad (27)$$

Next,

$$P\{\bigcup_{i=b_{n'}}^{n} \{|\bar{Y}_i^* - Q(W_i^*)| > \epsilon/4c_n\}\} \le P\{\bigcup_{i=1}^{n} \bigcup_{\ell=\bar{\lambda}_{b_n}}^{\infty} \{|Y(W_i^*, \ell) - Q(W_i^*)| > \epsilon/4c_n\}\}$$

$$\le n \sup_{w \in B} P\{\bigcup_{\ell=\bar{\lambda}_{b_n}}^{\infty} \{|Y(w, \ell) - Q(w)| > \epsilon/4c_n\}\}$$

where $Y(w, \ell) = \sum_{i=1}^{\ell} Y_i/\ell$ and the Y_i, $1 \le i \le n$, are iid random variables with distribution function F_w. Note that we used the fact that the merging in (9) is consistently used and that for all n, the W_n and W_i^n, $1 \le i \le I_n$, are pairwise unequal.

From lemmas 2 and 3 we know that for all n large enough

$$\sup_{w \in B} P_w\{\bigcup_{\ell=\bar{\lambda}_{b_n}}^{\infty} \{|Y(w, \ell) - Q(w)| > \epsilon/4c_n\}\} \le g_n$$

where

$$g_n = \begin{cases} 0 \text{ if the environment is } \mathcal{B} \\ K_1 c_n^t/\bar{\lambda}_{b_n}^{t-1} + K_2 e^{-K_3 \bar{\lambda}_{b_n}/c_n^2} \text{ if the environment is } \mathcal{E}_t \ (t\geq 2) \text{ and } \bar{\lambda}_{b_n}/c_n^2 \geq K_4 \\ K_5 \exp(-K_6 \bar{\lambda}_{b_n}/(K_7 c_n^2 + K_8 c_n)) \text{ if the environment is } \mathcal{J} \text{ and } \bar{\lambda}_{b_n}/c_n^2 \geq K_9. \end{cases}$$

and where $K_1, \ldots, K_9$ are positive constants that depend upon $\mathcal{E}$ and the parameters of the environment, e.g. L and t if $\mathcal{E}$ is $\mathcal{E}_t$ with parameter L, or L and σ^2 if the environment is $\mathcal{J}$. Let $d=\text{Min}(1/10; \theta/2)$ so that, after collecting bounds and resubstitution in (25), we obtain, for all n large enough,

$$P\{Q(W_n) > q_{min}+\epsilon\} \leq \exp(-\sum_{i=b_{n'}}^{n} \beta_i) + 2\exp(-d\sum_{i=b_{n'}}^{n} \alpha_i) + n g_n . \quad (28)$$

Clearly, (28) and (13-17) imply (18). The second part of the theorem follows from (28), (15'-17') and the Borel-Cantelli lemma [21]. Indeed, it is easy to check that for all $\epsilon>0$,

$$\sum_{n=1}^{\infty} P\{Q(W_n) > q_{min}+\epsilon\} < \infty$$

by a repeated use of the fact that for any sequence $\{a_n\}$ of nonnegative real numbers and any $r>0$, $a_n/\log n \to \infty$ if and only if $\sum_{n=1}^{\infty} n^r e^{-a_n} < \infty$. This concludes the proof of theorem 1.

Proof of theorem 3. Let $B=\{w_0, w_1\}$, let $Q(w_0)=0$, $Q(w_1)=1$ and let the environment be deterministic. Let G put mass 1/2 each at w_0 and w_1. Let $\gamma_n=0$ for all n and consider the algorithm described in section III with $\lambda_0=1$ (this is without loss of generality since the environment is deterministic). Theorem 3 is proved if we can show that

(i) If $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $P\{\bigcup_{k\geq n} \{Q(W_k^*)>\frac{1}{2}\}\}=1$ for all n

(ii) If $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $P\{Q(W_n)>\frac{1}{2}\} \geq \frac{1}{2}\exp(-\sum_{n=1}^{\infty} \alpha_n)$ for all n.

For (i), with arbitrary n, we argue as follows. Since $\sum_{k=n}^{\infty} \alpha_k = \infty$ for all n,

$$P\{\bigcup_{k\geq n} \{Q(W_k^*)>\tfrac{1}{2}\}\} \geq P\{\bigcup_{k\geq n} \{Z_k=1, W_k^*=w_1\}\}$$
$$= 1-P\{\bigcap_{k\geq n} [\{Z_k=0\}\cup\{Z_k=1, W_k^*=w_0\}]\}$$
$$= 1-\prod_{k\geq n} (P\{Z_k=0\}+P\{Z_k=1\}/2) \geq 1-\prod_{k\geq n} (\beta_k+\alpha_k/2)$$
$$= 1-\prod_{k\geq n} (1-\alpha_k/2) \geq 1-\exp(-\sum_{k=n}^{\infty} \alpha_k/2) = 1.$$

If $\sum_{n=1}^{\infty} \alpha_n = 0$, then, with probability one, $W_0=W_1=W_2=\ldots$ so that for all n, $P\{Q(W_n)>\frac{1}{2}\}=P\{Q(W_0)>\frac{1}{2}\}=\frac{1}{2}$. Next, let $\sum_{n=1}^{\infty} \alpha_n = A < \infty$. Then,

$$P\{Q(W_n)>\tfrac{1}{2}\} \geq P\{Q(W_0)>\tfrac{1}{2}\}.P\{\bigcap_{k=1}^{n} \{Q(W_k^*)=1\}\}$$
$$\geq \tfrac{1}{2} P\{\bigcap_{k=1}^{n} \{\{Z_k=1, W_k^*=w_1\}\cup\{Z_k=0\}\}\}$$
$$= \tfrac{1}{2}\prod_{k=1}^{n} (\beta_k+\alpha_k/2) = \tfrac{1}{2}\prod_{k=1}^{n} (1-\alpha_k/2)$$
$$\geq \tfrac{1}{2}\prod_{k=1}^{n} \exp(-\alpha_k/(2-\alpha_k)) \geq \tfrac{1}{2}\prod_{k=1}^{n} e^{-\alpha_k} \geq \tfrac{1}{2}e^{-A}$$

where we used the inequality $1-u\geq\exp(-u/(1-u))$ for $0\leq u<1$. This proves (ii). Next,

$$P\{Q(W_n^*)>\tfrac{1}{2}\} \geq \beta_n P\{Q(W_{n-1})>\tfrac{1}{2}\} \geq \beta_n e^{-A}/2$$

and

$$\liminf_n P\{Q(W_n^*)>\tfrac{1}{2}\} \geq e^{-A}/2$$

because the summability of the α_n implies that $\alpha_n \to 0$ and $\beta_n \to 1$. This and (i) show that it is impossible that $Q(W_n^*) \to q_{min}$ wpl.

Proof of the sufficiency of (22-24): We display a sequence $\{c_n\}$ with $c_n=n-b_n+1$ for all n such that (13-17) or (13-14, 15'-17') hold. Let $c_n \sim n^\gamma$ for some $0<\gamma<1$. Because $\beta<1$ and $\alpha<1$ we have that

$$\sum_{i=b_{n'}}^{n} \alpha_i \sim n^{\gamma-\alpha} \text{ and } \sum_{i=b_{n'}}^{n} \beta_i \sim n^{\gamma-\beta}.$$

Also, $\lambda_{b_n}/c_n^2 \log n \sim n^{\delta-2\gamma}/\log n$

and

$$n c_n^t/\lambda_{b_n}^{t-1} \sim n^{1+\gamma t-\delta(t-1)}.$$

If (22) holds, then we can find a $0<\gamma<1$ such that (13-17) is satisfied for type $\mathcal{E}_t$ environments with $t\geq 2$. Similarly, (24) is sufficient for (13-17) and for (15'-17') for type $\mathcal{J}$ environments. Finally, (23) is sufficient for (13-14, 15'-17') for type $\mathcal{E}_t$ environments.

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